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Future singularities of isotropic cosmologies

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Abstract

We show that globally and regularly hyperbolic future geodesically incomplete isotropic universes, except for the standard all-encompassing 'big crunch', can accommodate singularities of only one kind, namely, those having a non-integrable Hubble parameter, H. We analyze several examples from recent literature which illustrate this result and show that such behaviour may arise in a number of different ways. We also discuss the existence of new types of lapse singularities in inhomogeneous models, impossible to meet in the isotropic ones.

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1. Introduction

It has been pointed out very recently that finite-time, finite-Hubble parameter singularities can occur in the expanding direction of even the simplest FRW universes in general relativity and other metric theories of gravity and, although such behaviour depends on the details of the particular model, yet, it is met in a very wide and extremely varied collection of possible

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cosmologies (cf. refs. [1–30] for a partial list). These singularities cannot be accommodated by the usual singularity theorems [31], for their features arise from *necessary*, not sufficient, conditions for their existence while their character is somewhat milder than the standard, all-encompassing 'big crunch' singularities met in other model universes. This situation appears somewhat wanting as all known relevant models provide merely examples, not general theorems, of such a phenomenon.

What is the underlying reason for such a behaviour? In this paper, we show that the finite-time, finite-H singularities met in isotropic cosmologies satisfy all the assumptions of the completeness theorem of [32] *except* the one about an infinite proper-time interval of existence of privileged observers and because of this reason are geodesically incomplete. This leads to a classification of the possible types of singularities in the isotropic category based on necessary analytic conditions for their existence. We also briefly discuss how new types of such singularities may arise in more general inhomogeneous models.

2. Necessary conditions for future singularity formation

Although we shall focus exclusively on isotropic models, it is instructive to begin our analysis by taking a more general stance. Consider a spacetime $(\mathcal{V}, g)$ with $\mathcal{V} = \mathcal{M} \times \mathcal{I}$, $\mathcal{I} = (t_0, \infty)$, where $\mathcal{M}$ is a smooth manifold of dimension n and ${}^{(n+1)}g$ a Lorentzian metric which in the usual $n + 1$ splitting, reads

$$ {}^{(n+1)}g \equiv -N^2(\theta^0)^2 + g_{ij}\,\theta^i\theta^j, \quad \theta^0 = \mathrm{d}t, \quad \theta^i \equiv \mathrm{d}x^i + \beta^i \mathrm{d}t. \tag{2.1} $$

Here $N = N(t, x^i)$ is called the *lapse function*, $\beta^i(t, x^j)$ is called the *shift function* and the spatial slices $\mathcal{M}_t\,(= \mathcal{M} \times \{t\})$ are spacelike submanifolds endowed with the time-dependent spatial metric $g_t \equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We call such a spacetime a *sliced space* [33]. A sliced space is time-oriented by increasing t and we choose $\mathcal{I} = (t_0, \infty)$ because we shall study the future singularity behaviour of an expanding universe with a singularity in the past, for instance at $t = 0 < t_0$. However, since t is just a coordinate, our study could apply as well to any interval $\mathcal{I} \subset \mathbb{R}$.

A natural causal assumption for $(\mathcal{V}, g)$ is that it is *globally hyperbolic*. This implies the existence of a time function on $(\mathcal{V}, g)$. In a globally hyperbolic space the set of all timelike paths joining two points is compact in the set of paths, and spacetime splits as above with each spacelike slice $\mathcal{M}_t$ a Cauchy surface, i.e., such that each timelike and null path without end points cuts $\mathcal{M}_t$ exactly once [34].

We say that a sliced space has *uniformly bounded lapse* if the lapse function N is bounded below and above by positive numbers N_m and N_M,

$$ 0 < N_m \leq N \leq N_M. \tag{2.2} $$

A sliced space has *uniformly bounded shift* if the g_t norm of the shift vector β, projection on the tangent space to $\mathcal{M}_t$ of the tangent to the lines $\{x\} \times \mathcal{I}$, is uniformly bounded by a number B.

A sliced space has *uniformly bounded spatial metric* if the time-dependent metric $g_t \equiv g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$ is uniformly bounded below for all $t \in \mathcal{I}$ by a metric $\gamma = g_{t_0}$, that is there exists a number $A > 0$ such that for all tangent vectors v to $\mathcal{M}$ it holds that

$$A\gamma_{ij}v^i v^j \leq g_{ij}v^i v^j. \tag{2.3}$$

A sliced space $(\mathcal{V}, g)$ with uniformly bounded lapse, shift and spatial metric is called *regularly hyperbolic*.

Denoting by ∇N the space gradient of the lapse N, by $K_{ij} = -N\Gamma^0_{ij}$ the extrinsic curvature of $\mathcal{M}_t$, and by $|K|^2_g$ the product $g^{ai}g^{bj}K_{ab}K_{ij}$, we have the following theorem which gives sufficient conditions for geodesic completeness [32]:

Theorem 2.1. *Let $(\mathcal{V}, g)$ be a sliced space such that the following assumptions hold:*

C1 *$(\mathcal{V}, g)$ is globally hyperbolic.*
C2 *$(\mathcal{V}, g)$ is regularly hyperbolic.*
C3 *For each finite t_1, the space gradient of the lapse, $|\nabla N|_g$, is bounded by a function of t which is integrable on $[t_1, +\infty)$.*
C4 *For each finite t_1, $|K|_g$ is bounded by a function of t which is integrable on $[t_1, +\infty)$.*

Then $(\mathcal{V}, g)$ is future timelike and null geodesically complete.

It is known that in a regularly hyperbolic spacetime, condition C1 is in fact *equivalent* to the condition that each slice of $(\mathcal{V}, g)$ is a complete Riemannian manifold, cf. [33]. The completeness Theorem 2.1 gives sufficient conditions for timelike and null geodesic completeness and therefore implies that the negations of each one of conditions C1–C4 are *necessary* conditions for the existence of singularities (while, the singularity theorems (cf. [31]) provide sufficient conditions for this purpose). Thus, Theorem 2.1 is precisely what is needed for the analysis of models with a big rip singularity inasmuch as these models require *necessary* conditions for future singularities.

Now for an FRW metric, $N = 1$ and $\beta = 0$, and so condition C2 is satisfied provided the scale factor $a(t)$ is a bounded from below function of the proper time t on $\mathcal{I}$. Condition C3 is trivially satisfied and this is again true in more general homogeneous cosmologies where the lapse function N depends only on the time and not on the space variables. In all these cases its space gradient, $|\nabla N|_g$, is zero. Condition C4 is the only other condition which can create a problem. For an isotropic universe $|K|_g{}^2 = 3(\dot{a}/a)^2 = 3H^2$, and so we conclude that FRW universes in which the scale factor is bounded below can fail to be complete only when C4 is violated. This can happen in only one way: there is a finite time t_1 for which H fails to be integrable on the time interval $[t_1, \infty)$. Since this non-integrability of H can be implemented in different ways, we arrive at the following result for the types of future singularities that can occur in isotropic universes.

Theorem 2.2. *Necessary conditions for the existence of future singularities in globally hyperbolic, regularly hyperbolic FRW universes are:*

S1 *For each finite t, H is non-integrable on $[t_1, t]$, or,*
S2 *H blows up in a finite time, or,*

S3 *H is defined and integrable (that is bounded, finite) for only a finite proper time interval.*

When does Condition S1 hold? It is well known that a function $H(\tau)$ is integrable on an interval $[t_1, t]$ if $H(\tau)$ is defined on $[t_1, t]$, is continuous on (t_1, t) and the limits $\lim_{\tau\to t_1^+} H(\tau)$ and $\lim_{\tau\to t^-} H(\tau)$ exist. Therefore there are a number of different ways which can lead to a singularity of the type S1 and such singularities are in a sense more subtle than the usual ones predicted by the singularity theorems. For instance, they may correspond to 'sudden' singularities (see [19] for this terminology) located at the right end (say t_s) at which H is defined and finite but *the left limit*, $\lim_{\tau\to t_1^+} H(\tau)$, may fail to exist, thus making H non-integrable on $[t_1, t_s]$, for *any* finite t_s (which is of course arbitrary but fixed from the start). We shall see examples of this behaviour in the next Section. Condition S2 leads to what is called here a blow-up singularity corresponding to a future singularity characterized by a blow-up in the Hubble parameter. Note that S1 is not implied by S2 for if H blows up at some finite time t_s after t_1, then it may still be integrable on $[t_1, t]$, $t_1 < t < t_s$. Condition S3 also may lead to a big-rip type singularity, but for this to be a genuine type of singularity (in the sense of geodesic incompleteness) one needs to demonstrate that the metric is non-extendible to a larger interval.

3. Examples

We give below some representative examples to illustrate the results of the previous Section. An example of a blow-up singularity is the recollapsing flat FRW model filled with dust and a scalar field with a 'multiple' exponential potential considered in [18]. These authors choose the potential to be of the form $V(\phi) = W_0 - V_0 \sinh(\sqrt{3/2}\kappa\phi)$ where W_0 and V_0 are arbitrary constants and $\kappa = \sqrt{8\pi G_N}$, and split the scale factor according to the transformation $a^3 = xy$. Here one sets $x = C[\exp \chi_1 \cos \chi_2 + \exp(-\chi_1) \cos \chi_2]$, $y = C[\exp \chi_1 \sin \chi_2 + \exp(-\chi_1) \sin(-\chi_2)]$ with $\chi_1 = w_1(t - t_0)$, $\chi_2 = w_2(t - T_0)$, where $C > 0$, t_0 an arbitrary constant, T_0 is the 'initial' time and w_1, w_2 positive parameters such that $w_1^2 - w_2^2 = 3/4\kappa^2 W_0$, $2w_1w_2 = 3/4\kappa^2 V_0$. Using the results of Section 2, we can immediately see why this model has such a singularity (in [18] this was proved by different methods). For large and positive values of the time parameter t the scale factor becomes $a = C^{2/3} \exp(2/3\chi_1) \cos^{1/3} \chi_2 \sin^{1/3} \chi_2$, which is obviously divergent. We then find $|K|_g$ to be essentially (that is except unimportant constants) proportional to $w_2(\cot \chi_2 - \tan \chi_2)/3$, and so the extrinsic curvature blows up as t tends to the *finite* time value $\pi/(2w_2) + T_0$. This is a blow-up singularity in the sense of Condition S2 of Section 2.

Another example of a future blow-up singularity is provided by the 'phantom' cosmologies (see relevant references in [19]). In all these models different methods are used, depending each time on the analysis of the field equations of each phantom model, to prove the existence of such future singularities. For instance, in [17] we meet a flat FRW universe with a minimally coupled scalar field ϕ in Einstein's gravity with the equation of state $p = w\rho$, with $w < -1$ (phantom dark energy). In this model the scale factor takes the form

$$a = \left[{a_0}^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t - t_0) \right]^{(2/3(1+w))}, \tag{3.1}$$

where $A = 8\pi GC/3$ and C an integration constant. Hence, the extrinsic curvature becomes

$$|K|_g^2 = 3A\left[{a_0}^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t-t_0)\right]^{-2}, \tag{3.2}$$

and so when $w < -1$ and $t = t_0 + 2/[3\sqrt{A}(|w|-1)]a_0^{3(1-|w|)/2}$, the extrinsic curvature diverges thus causing a blow-up singularity. Similar behaviour is found in other singular phantom cosmologies, for instance those of refs. [1,9,12,16,21], namely, there is *some* finite time t_f at which the Hubble parameter blows up.

Had the evolution been characterized by an integrable H *for every t*, then we would expect from Theorem 2.1 all these models to be timelike and null geodesically complete. An example illustrating this is given by the flat FRW universe filled with phantom dark energy which behaves simultaneously as Chaplygin gas studied in [25] (see also [13]). Here the phantom dark energy component satisfies the equation of state $p = w\rho$ with $w < -1$, as well as the equation of state of a Chaplygin gas $p = -A/\rho$, where A is a positive constant (similar results will hold when $w \in (-1, -1/3)$ -k-essence models, see, e.g., [17]). Then integrating the continuity equation $\dot{\rho} = -3\dot{a}(\rho + p)/a$, we find

$$\rho^2(t) = A + (\rho_0^2 - A)\left(\frac{a_0}{a(t)}\right)^6, \tag{3.3}$$

or, using the two equations of state and substituting to (3.3) leads to

$$\rho(t) = \rho_0\left[-w_0 + (1+w_0)\left(\frac{a_0}{a(t)}\right)^6\right]^{1/2}, \tag{3.4}$$

where $A = -w_0{\rho_0}^2$, $w_0 < -1$. Finally, the solution for the scale factor reads $a(t) = Ce^{C_1(t-t_0)}$, and so the Hubble parameter in this case is constant, $H = C_1$. This non singular behavior is due to the fact that all the conditions of the completeness Theorem 2.1 are met.

We now move on to an example of a big rip singularity in the sense Condition S1 of Section 2. Barrow in [19] rightly calls such singularities 'sudden'. He considers an expanding FRW model with a fluid that satisfies the energy conditions $\rho > 0$ and $\rho + 3p > 0$, and shows that the special solution

$$a(t) = 1 + \left(\frac{t}{t_\mathrm{s}}\right)^q (a_\mathrm{s} - 1) + \left(1 - \frac{t}{t_\mathrm{s}}\right)^n, \tag{3.5}$$

with $1 < n < 2$, $0 < q \leq 1$ and $a_\mathrm{s} \equiv a(t_\mathrm{s})$, exists, as a smooth solution, only on the interval $(0, t_s)$, while a_s and $H_\mathrm{s} \equiv H(t_\mathrm{s})$ are finite at the right end. Note that setting $A = q(a_\mathrm{s} - 1)/t_\mathrm{s}^q$, $B = n/t_\mathrm{s}^n$, we find that $\dot{a} = At^{q-1} + B(t_\mathrm{s} - t)^{n-1}$ which means that, unless $q = 1$, $\dot{a}$ blows up as $t \to 0$, making H continuous only on $(0, t_\mathrm{s})$. Also $a(0)$ is finite and we can extend H and define it to be finite also at 0, $H(0) \equiv H_0$, so that H is defined on $[0, t_\mathrm{s}]$. However, since $\lim_{t\to 0^+} H(t) = \pm\infty$, we conclude that this model universe implements exactly Condition S1 of the previous Section and thus H is non-integrable on $[0, t_\mathrm{s}]$, t_s arbitrary. This is a big rip singularity characterized by the fact that as $t \to t_\mathrm{s}$ one obtains $\ddot{a} \to -\infty$. Then using the field equation we see that this is really a divergence in the pressure,

$p/2 \to -\ddot{a}/a - H^2/2 - k/(2a^2)$, and so $p \to \infty$. In particular, we cannot have in this universe a family of privileged observers each having an infinite proper time and finite H.

Another way to see this result in the particular case when $\rho > 0$ and $p \geq 0$, is to use the following result from [31], pp. 141–2: in an FRW universe with $\rho > 0$ and $p \geq 0$, given any vector X at any point q, the geodesic $\gamma(\upsilon)$ which passes through the point $q = \gamma(0)$ in the direction of X, is such that either

- $\gamma(\upsilon)$ can be extended to arbitrary values of its affine parameter υ, *or*
- there is some value $\upsilon_0 > 0$ such that the scalar invariant $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is unbounded on $\gamma([0, \upsilon])$.

Therefore, under the assumptions of [19], the invariant $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is calculated to be

$$\frac{12}{a^4} + \frac{24\dot{a}^2}{a^4} + \frac{12\dot{a}^4}{a^4} + \frac{12\ddot{a}}{a^3} + \frac{12\dot{a}^2\ddot{a}}{a^3} + \frac{12\ddot{a}^2}{a^2}, \tag{3.6}$$

and since $a \to a(t_s)$, $H(t) \to H_s$, $p(t) \to \infty$, $\ddot{a}/a \to -\infty$ as $t \to t_s$, we see that $(R_{\alpha\beta} - 1/2Rg_{\alpha\beta})(R^{\alpha\beta} - 1/2Rg^{\alpha\beta})$ is unbounded at t_s. Hence, we find that this spacetime is geodesically incomplete.

Further, Carroll et al. in [14] provide a different model leading, however, to a similar future big rip singularity. They start from a theory of gravity with lagrangian density $L = R + \alpha R^{-2} + L_{\mathrm{mat}}$ and conformally transform to an Einstein frame to get an equivalent theory which is described as general relativity coupled to a system comprised by the scalar field resulting from the conformal transformation and the conformally transformed matter. Since the two matter components in the Einstein frame are not separately conserved, there is now a non-trivial scalar field-matter interaction which manifests itself with new terms appearing in the field equations. It then follows that as $\phi \to 0$, the time derivative of the extrinsic curvature blows up, $\dot{H} \to \infty$, while H itself is finite at $\phi = 0$ (cf. [14]). This is clearly a big-rip type singularity.

A different example of a big rip singularity is given by Borde et al. in [3]. They showed that in inflating spacetimes just a bounded averaged-out Hubble function is necessary to produce a singularity with *finite H*. Why do they obtain such a behaviour? To answer this question we again use our Theorem 2.2 and conclude that their condition given by Eq. (11) in [3] is precisely the negation of condition C4 of Theorem 2.1 above. In effect, what these authors do is to find that a necessary condition for a singularity in the spacetimes considered is that H be finite on a *finite* time interval. One recognizes that under their assumptions, privileged observers cannot exist in inflating universes for an infinite proper time, because had such observers existed, Theorem 2.1 would then imply that these inflating spacetimes are timelike and null geodesically complete. More precisely, in [3] one starts with a flat FRW model, $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\vec{x}^2$, and considers all quantities along a null geodesic with affine parameter λ. Since the model is conformally Minkowski we have $\mathrm{d}\lambda \propto a(t)\,\mathrm{d}t$, or, $\mathrm{d}\lambda = a(t)\,\mathrm{d}t/a(t_s)$, so that $\mathrm{d}\lambda/\mathrm{d}t = 1$ for $t = t_s$, where t_s is a finite value of time. One then defines

$$H_{\mathrm{av}} = \frac{1}{\lambda(t_s) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_s)} H(\lambda)\,\mathrm{d}\lambda, \tag{3.7}$$

and so if $H_{\mathrm{av}} > 0$ one finds

$$0 < H_{\mathrm{av}} = \frac{1}{\lambda(t_{\mathrm{s}}) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_{\mathrm{s}})} H(\lambda)\,\mathrm{d}\lambda$$
$$= \frac{1}{\lambda(t_{\mathrm{s}}) - \lambda(t_i)} \int_{a(t_i)}^{a(t_{\mathrm{s}})} \frac{\mathrm{d}a}{a(t_{\mathrm{s}})} \leq \frac{1}{\lambda(t_{\mathrm{s}}) - \lambda(t_i)}. \tag{3.8}$$

This shows that the affine parameter must take values only in a finite interval which implies geodesic incompleteness. A similar proof is obtained for the case of a timelike geodesic. Notice that condition (3.8) holds if and only if condition C4 of Theorem 2.1 is valid only for a finite interval of time, thus leading to incompleteness according to Theorem 2.2. A similar bound for the Hubble parameter is obtained in [3] for the general case, and one therefore concludes that such a model must be geodesically incomplete.

A relaxation of the requirement that H be finite for only a finite amount of proper time leads, as expected by Theorem 2.1, to singularity-free inflationary models evading the previously encountered singularity behaviour. Such models have been considered in [15] (see also [4,8]). It is particularly interesting in our context to see why such behaviour actually occurs: in [15], the scale factor is assumed to be bounded below by a constant a_i (so that this universe is regularly hyperbolic in our notation), and its form is given by (these authors also take this to be qualitatively true in more general models),

$$a(t) = a_i \left[1 + \exp\left(\frac{\sqrt{2}t}{a_i}\right)\right]^{1/2}. \tag{3.9}$$

One then finds that H is not only bounded as $t \to -\infty$, but actually becomes zero asymptotically. Thus, by Theorem 2.1 this universe is geodesically complete. It is instructive, however, to notice that a slight change in the inflationary solution can drastically alter this behaviour. Consider for example the old quasi-exponential models first considered by Starobinski (see [35] and references therein), where the scale factor is given by $a = c_0 \exp[c_1 t + c_2 t^2]$, with the c_i's constants. Then we see that $H = c_1 + 2c_2 t$ and so it blows up to $\pm\infty$ as $t \to \pm\infty$ depending on the sign of the constant c_2. It is interesting to further notice (cf. [35]) that this solution is an attractor of all homogeneous, isotropic solutions of higher order gravity theories and so we may conclude that the blow up behaviour leading to a singular regime is perhaps a more generic feature than completeness in such contexts.

Examples of big rip as well as geodesically complete types of evolution are also met in many brane cosmologies, cf. [5–7,10,11,26]. The singularities in such models are all characterized by the fact that there exists no admissible slicing with an infinite proper time of privileged observers and the Hubble parameter remains finite but only for this finite interval of proper time and cannot be defined beyond that interval (cf. [5]), and hence in accordance with Theorem 2.2 these spacetimes must be geodesically incomplete.

However, geodesically complete solutions also exist and such phases in brane evolution are described in detail by the series of brane models studied in [10]. The main idea is rather simple, cf. [7,26]. We start with a flat 3-brane filled with ordinary matter as well as phantom dark energy embedded in a five-dimensional bulk. This model goes through a series of cycles of finite accelerating expansion and contraction. At the end of each cycle it bounces avoiding in this way a singular behavior. In the expansion phase the phantom dark energy component increases and drives cosmic acceleration. This rapid acceleration tears apart any bound structure produced during expansion and towards the end of the expanding phase phantom dark energy has become so high that modifications of the Friedman equation become important. The modified Friedman equation on the 3-brane is

$$H^2 = \frac{\Lambda_4}{3} + \frac{8\pi}{3M_{\rm p}^2}\rho + \epsilon\left(\frac{4\pi}{3M_5^3}\right)^2\rho^2 + \frac{C}{a^4}, \tag{3.10}$$

where C is an integration constant, ϵ corresponds to the metric signature of the extra dimension and Λ_4 is the cosmological constant on the brane. Assuming that C/a^4 is negligible and $\epsilon < 0$, we have for the critical brane case ($\Lambda_4 = 0$)

$$H^2 = \frac{8\pi}{3M_{\rm p}^2}\left(\rho - \frac{\rho^2}{2|\sigma|}\right), \tag{3.11}$$

hence, $H = 0$ when $\rho_{\rm bounce} = 2|\sigma|$ where σ is the tension of the brane. At this scale the model will turn around and start to contract. During contraction the phantom component will decrease but at late times, when the model becomes matter or radiation dominated, the energy density will be very high and modifications of the Friedman equation will again become important. When the critical value of the density is reached the model will bounce and start to expand. Using the assumptions of this model, we see from Eq. (3.11) that the Hubble parameter remains eternally finite and also all other assumptions of Theorem 2.1 are satisfied. Therefore this universe is geodesically complete.

4. Discussion

The main result of this paper is the recognition that future big rip singularities occur because there is a finite time, say at 0, such that the Hubble parameter is not integrable on $[0, \infty)$. In such universes privileged observers cannot exist for an infinite proper time starting from 0. We know from Theorem 2.1 (cf. [32]) that if such observers existed for an infinite proper time, then such a universe would be timelike and null geodesically complete. This condition is not satisfied in recent models with big rip singularities and therefore such universes may not be complete. In fact, examples show, cf. refs. [3,14,19], that such spaces are necessarily singular.

It is interesting that isotropic models coming from completely different motivations reveal their tendencies to have future singularities (that is future geodesic incompleteness) for exactly one the same reason, namely the non-integrability of H. Of course, the fact that this condition can be sustained in different models stems from the different ways these models are constructed and the specific features they share. It may be supported by the presence of a scalar field inducing a novel interaction with matter as in the case of $f(R)$ theory considered in ref. [14], or lead to a divergence in the fluid pressure and a corresponding one in $\ddot{a}$ as in [19], or in the inflationary character of the particular model as in [3], etc.

We have further shown that when we restrict attention to homogeneous and isotropic universes, these non-integrable-H singularities are the only types of future big-rip singularities which occur. This leads us to the following question: Is it possible to find a globally hyperbolic, regularly hyperbolic inhomogeneous cosmology which satisfies condition C4 of Theorem 2.1 but *not* C3? Since $|\nabla N|_g$ is now a function not only of the time but also of the space variables it is not automatically zero. If true, this effect will lead, in addition to the ones already present in the isotropic case (i.e., those having an integrable $|\nabla N|_g$ but failing C4), to two new types of *lapse singularities* both not satisfying condition C3. Namely, that which has a diverging $|\nabla N|_g$ in a finite time corresponding to a *blow-up lapse singularity* and secondly, that with $|\nabla N|_g$ finite only for a finite interval of proper time, a *big-rip lapse singularity*. The latter singularity, if it exists, will necessarily have several qualitative features distinct from the corresponding blow-up or big-rip (extrinsic) curvature singularities discussed here. In turn, these will not anymore be the only available types of singularity in the inhomogeneous category, as indeed they are for isotropic universes.

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